Generalizing the relative gap for measuring the convergence of a logit-based user-equilibrium assignment

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Introduction

Background

In the case of a user-equilibrium assignment (UE), sufficient convergence of the iterative algorithm is regarded as essential for the stability of the obtained set of traffic flows. With the relative gap (RelGap-DUE), there is a commonly used approach for measuring convergence. In its current formulation, it is, however, only properly applicable for a **deterministic user-equilibrium (DUE)** assignment. In this paper, a generalized formulation of the relative gap is introduced that can also be used in a **logit-based stochastic user-equilibrium (MNL-SUE)** environment.

Objectives

- **Develop a convergence measure** for a logit-based user-equilibrium assignment (MNL-SUE) that constitutes a generalization of the relative gap (RelGap-DUE).
- **Keep the formulation and value range** as close as possible to those of 'original' relative gap (RelGap-DUE).

A Way to Derive the 'Original' Relative Gap

We start by examining a way to derive the relative gap (RelGap-DUE) from an mathematical optimization problem by BECKMANN et al. [1]. The optimization problem is only solved by traffic flows d_{odr} equivalent to the traffic flows under DUE conditions.

Optimization problem (DUE)

$$\min_{d_{odr}} z(d_{odr}) = \sum_s \int_0^{\sum_{od}\sum_r d_{odr}\delta_{s,r}^{od}} t_s(\omega)\mathrm{d}\omega \qquad \text{subject to } d_{od} = \sum_r d_{odr} \text{ and } d_{odr} \geq 0$$

First-order (KKT) conditions	μ, λ (KKT) multipliers
$\nabla_{d_{odr}} L(d_{odr}, \lambda_{od}, \mu_{odr}) = 0$	$\mu_{odr} = t_{odr} - \lambda_{od}$
$d_{od} - \sum_r d_{odr} = 0$	$\lambda_{od} = \min_r t_{odr}$
$-d_{odr} \leq 0$	
$\mu_{odr} \geq 0$	
(Complementary slackness condition) $d_{odr}\mu_{odr} = 0$	

For our proposes, we are interested in the first-order (Karush-Kuhn-Tucker, KKT) conditions for a solution of the problem (which are also sufficient conditions since we are dealing with a strictly convex optimization problem). To derive the relative gap (RelGap-DUE) based on the first-order conditions, we proceed as follows:

1. **We form the gradient of the Lagrangian $L(d_{odr}, \lambda_{od}, \mu_{odr})$ with respect to** d_{odr} and solve the equation of $\nabla_{d_{odr}} L(d_{odr}, \lambda_{od}, \mu_{odr}) = 0$ towards the μ_{odr} multiplier to determine an expression based on travel times t_{odr} for the multiplier.

2. **Since the other KKT multiplier λ_{od} is still part of this expression**, it is in turn expressed in terms of travel times by introducing λ_{od} as the minimum travel time $\min_r t_{odr}$ for an origin-destination pair. This is possible since at the equilibrium point the travel time on all used routes between a given origin and destination is equal to the minimum travel time between that origin-destination pair.

3. **In order to form the relative gap (RelGap-DUE)**, we now consider the complementary slackness condition where we replace μ_{odr} with its equivalent expression in terms of travel times. Then we form the sums over all routes and origin-destination pairs, yielding the gap formulation for DUE. Introducing the total travel time as a reference value then yields the known relative gap (RelGap-DUE) formulation.

GAP_{DUE}	$RelGAP_{DUE}$
$\sum_{od}\sum_r d_{odr}\left(t_{odr} - \min_r t_{odr}\right)$	$\dfrac{\sum_{od}\sum_r d_{odr}\cdot\left(t_{odr} - \min_r t_{odr}\right)}{\sum_{od}\sum_r d_{odr}\cdot t_{odr}}$

Note that we can ignore the other first-order conditions due to the nature of solution algorithms such as Franke-Wolfe or MSA that only select feasible descent directions.

Development of a Relative Gap for Logit-based SUE (and DUE)

The idea is now to form a convergence measure for MNL-SUE by looking at the complementary slackness condition and the KKT multipliers in the stochastic case. For that, we consider the optimization problem for MNL-SUE by FISK [2].

Optimization problem (MNL-SUE)

$$\min_{d_{odr}} z(d_{odr}) = \sum_s \int_0^{\sum_{od}\sum_r d_{odr}\delta_{s,r}^{od}} t_s(\omega)\mathrm{d}\omega + \frac{1}{\beta}\sum_{od}\sum_r d_{odr}\ln d_{odr} \qquad \text{subject to } d_{od} = \sum_r d_{odr} \text{ and } d_{odr} \geq 0$$

First-order (KKT) conditions	μ, λ (KKT) multipliers
$\nabla_{d_{odr}} L(d_{odr}, \lambda_{od}, \mu_{odr}) = 0$	
$d_{od} - \sum_r d_{odr} = 0$	$\mu_{odr} = t_{odr} + \frac{1}{\beta}\ln d_{odr} + \frac{1}{\beta} - \lambda_{od}$
$-d_{odr} \leq 0$	$\lambda_{od} = E\left[\min_r t_{odr}\right] + \frac{1}{\beta}\ln d_{od} + \frac{1}{\beta}$
$\mu_{odr} \geq 0$	where
(Complementary slackness condition) $d_{odr}\mu_{odr} = 0$	$E\left[\min_r t_{odr}\right] = -\frac{1}{\beta}\ln\left(\sum_r \exp^{(-\beta\cdot t_{odr})}\right)$

In comparison to the optimization problem for DUE, the objective function in the case of MNL-SUE is extended by a 'stochastic' or rather entropic term while the constraints remain unchanged. To develop a relative gap for MNL-SUE, we proceed as follows:

1. **We determine an expression for the μ_{odr} multiplier** in the same way as in the DUE case. Due to the additional 'stochastic' term in the objective function, however, the μ_{odr} multiplier is dependent on the travel times t_{odr} and the traffic flows d_{odr}.

2. **To express the λ_{od} multiplier in the MNL-SUE case in terms of travel times and traffic flows**, we introduce the multiplier as an expression without route enumeration that at the equilibrium point is equivalent to the term $t_{odr} + \frac{1}{\beta}\ln d_{odr} + \frac{1}{\beta}$ for all used routes. In comparsion to RelGap-DUE, the minimum travel time $\min_r t_{odr}$ for an origin-destination pair can here be interpreted as the expected value of the perceived travel time $E\left[\min_r t_{odr}\right]$ (see SHEFFI [3]) for an origin-destination pair as travelers are trying to minimize their perceived travel time and not the measurable travel time.

3. **We use the complementary slackness condition again as a basis** to form the gap for MNL-SUE. By substituting the new expression of λ_{od} into the new expression of μ_{odr} and the combined expression into the complementary slackness condition, we obtain a gap formulation for MNL-SUE. Introducing the total travel time again as a reference value then yields a relative gap formulation for MNL-SUE.

GAP_{MNL}	$RelGAP_{MNL}$
$\sum_{od}\sum_r d_{odr}\left(t_{odr} + \frac{1}{\beta}\ln\frac{d_{odr}}{d_{od}} - E\left[\min_r t_{odr}\right]\right)$	$\dfrac{\sum_{od}\sum_r d_{odr}\cdot\left(t_{odr} + \frac{1}{\beta}\ln\left(\frac{d_{odr}}{d_{od}}\right) - E\left[\min_r t_{odr}\right]\right)}{\sum_{od}\sum_r d_{odr}\cdot t_{odr}}$

When comparing both relative gap approaches, the more universal character of the relative gap for MNL-SUE is easily noticeable. Just as the DUE assignment is a specific case of the MNL-SUE assignment, the relative gap for DUE is a specific version of the relative gap for MNL-SUE. The 'switch' between both relative gaps is the β parameter.

- **For measuring the convergence of a DUE assignment**, the parameter β needs to be set at ∞. As a result, the 'stochastic' term $\frac{1}{\beta}\ln\left(\frac{d_{odr}}{d_{od}}\right)$ becomes zero and the 'expected perceived travel time' $E\left[\min_r t_{odr}\right]$ equals the minimum travel time $min_r\, t_{odr}$. Hence, all stochastic elements are filtered out of the gap function and the relative gap indicates the convergence of DUE, an equilibrium based on all travelers objectively choosing a route with minimum travel time.
- **Measuring the convergence of MNL-SUE works by** setting the β parameter in the gap function as the scaling parameter β that is used in the MNL-SUE case for increasing or decreasing subjectivity concerning route choice. Decreasing β from ∞ towards zero results in travelers increasingly choosing routes based on their perceived utility rather than their objective travel time. Thus there occurs a shift from measurable route travel time to subjective route utility as a route choice criterion.

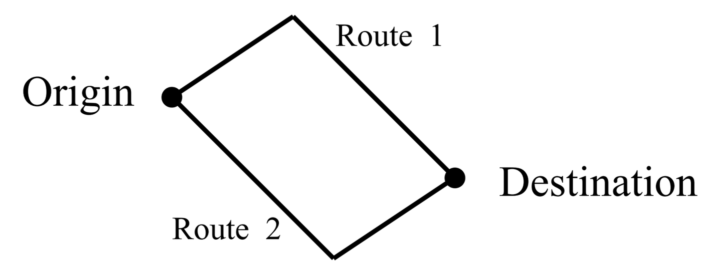
In summary, the relative gap (RelGap-MNL) works for DUE and MNL-SUE with

$$\text{DUE}:\ \beta = \infty \qquad \text{MNL-SUE}:\ \beta = Parameter$$

Numerical Example

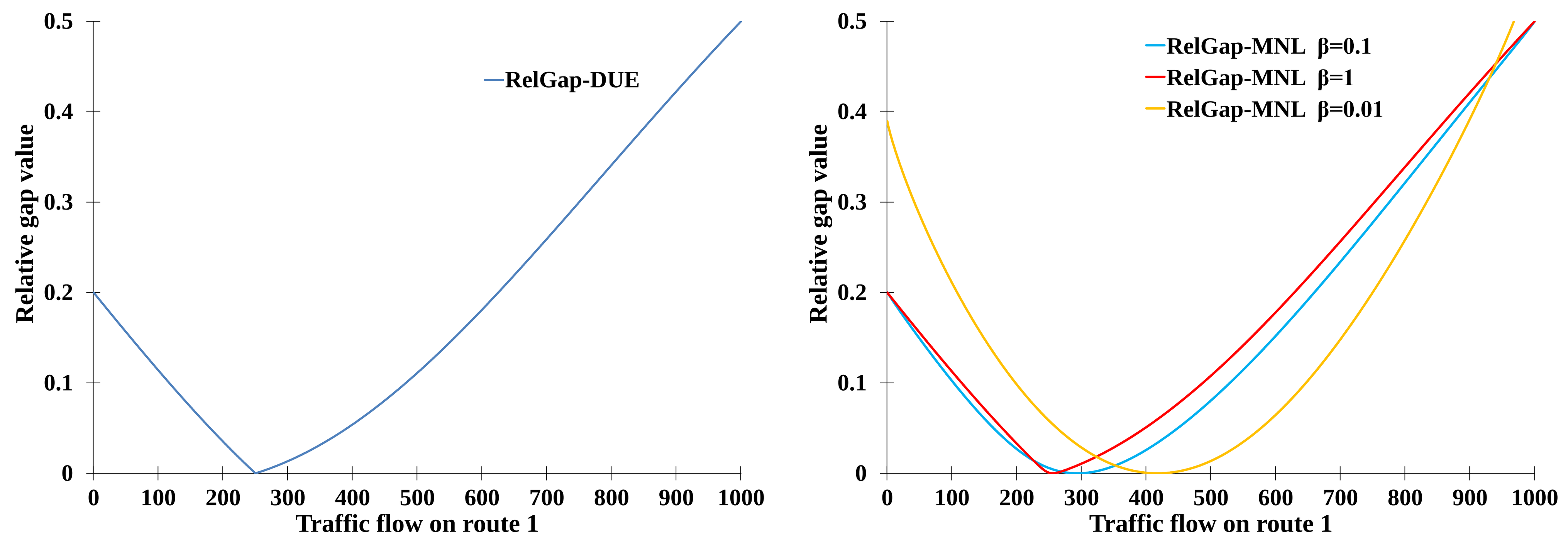
Example network

We use the small, two route network in the figure below as a basis for the following numerical examples. For further simplicity, each route only consists of one link, and we assume a linear relationship between traffic flow and travel time. The traffic flow for the origin-destination pair is assumed to be $d = 1000$ and the flow conservation constraint $d = d_1 + d_2$ holds at all times.



Behavior of RelGap-MNL for Different β

For $\beta = 1$, the graph of RelGap-MNL (Figure b) behaves similarly to the graph of RelGap-DUE (Figure a). This concurs with our suggestion to use RelGap-MNL with $\beta = \infty$ for a DUE assignment. However, if we decrease β towards zero, the behavior of the graph of RelGap-MNL changes as the minimum point moves along the x-axis, increasing the traffic flow on route 1 towards a new optimal solution of MNL-SUE traffic flows.



(a) DUE

(b) MNL-SUE ($\beta = 1$, $\beta = 0.1$, $\beta = 0.01$)

In the process, the graph of RelGap-MNL also gets a more parabolic form as the importance of the stochastic elements increases by decreasing β. As this creates a lower relative gap for solutions close to the optimal solution, it indicates that a set threshold for RelGap-MNL should depend in some way on the β parameter that is being used.
Considering the paper of BOYCE et al. [4], where a value for RelGap-DUE (RelGap-MNL where $\beta = \infty$) is suggested, further studies are needed to define RelGap-MNL values that ensure acceptable levels of convergence, examining the idea of a range of required thresholds depending on different β.

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