Introduction

Background

In the case of a user-equilibrium assignment (UE), sufficient convergence of the iterative algorithm is regarded as essential for the stability of the obtained set of traffic flows. With the relative gap (RelGap-DUE) as a common and convenient measure for measuring convergence, in the current formulation, it is, however, only properly applicable to deterministic user-equilibrium (DUE) assignment. In this paper, a generalized formulation of the relative gap that is introduced is used to be also in use in a hybrid stochastic-user-equilibrium (MNL-SUE) environment.

Objectives

1. Develop a convergence measure for a logit-based user-equilibrium assignment (MNL-SUE) that constitutes a generalization of the relative gap (RelGap-DUE).
2. Keep the formulation and value range as close as possible to those of 'original' relative gap (RelGap-DUE).

A Way to Derive the "Original" Relative Gap

We start by examining a way to derive the relative gap (RelGap-DUE) from an optimization problem described by Beckman et al. [1]. The optimization problem is only possible if the traffic flows are under (UE) conditions.

Optimization problem (DUE)

\[
\begin{align*}
\min & \quad \sum_{r} t_{rd} x_{rd} \quad \text{s.t.} \quad \sum_{d} x_{rd} = d_{r} \quad \forall r, \quad \sum_{r} x_{rd} = d_{d} \quad \forall d, \quad x_{rd} \geq 0 \quad \forall r, \quad d
\end{align*}
\]

We form the gradient of the Lagrangean \( \lambda_{td} \) with respect to \( x_{td} \) and solve the equation of \( \partial L / \partial x_{td} = \lambda_{td} - \beta_{td} \) for the relative gap (RelGap-DUE) to determine an expression based on travel times for the multiplier.

Since the other KKT multiplier \( \mu_{odr} \) is part of this expression, it is not expressed in terms of travel times by introducing the equation as \( \partial L / \partial x_{rd} = \mu_{odr} = \beta_{odr} \). In all assignments, in comparison to RelGap-DUE, the minimum travel time \( t_{od} \) for an origin destination pair can be interpreted as the expected value of the perceived travel time \( E(t_{od}) \) for an origin destination pair as travelers are trying to realize their perceived travel time and not the measurable travel time.

In order to form the relative gap (RelGap-DUE), we now consider the complementary slackness condition where we replace \( \mu_{odr} \) with \( t_{od} \) instead of travel times in terms of the assignment problem. Hereby, the components \( \mu_{odr} \) and \( t_{od} \) are located in the same row of matrices and origin destination pairs.

In summary, the relative gap (RelGap-DUE) for DUE and MNL-SUE with \( \beta = 0.01 \) can be derived.

The idea is now to form a convergence measure for MNL-SUE by looking at the complementary slackness condition and the KKT multipliers in the stochastic case. For that, we consider the optimization problem for MNL-SUE by fixing \( \beta = 0.01 \).

Development of a Relative Gap for Logit-based SUE (and DUE)

Optimization problem (MNL-SUE)

\[
\begin{align*}
\min & \quad \sum_{r} \left( \sum_{d} x_{rd} \ln \left( \frac{E(t_{rd})}{t_{rd}} \right) \right) \quad \text{s.t.} \quad \sum_{d} x_{rd} = d_{r} \quad \forall r, \quad \sum_{r} x_{rd} = d_{d} \quad \forall d, \quad x_{rd} \geq 0 \quad \forall r, \quad d
\end{align*}
\]

For measuring the convergence of a DUE assignment, the non-uniform characteristic of the relative gap (RelGap-DUE) is nicely visible. Just as the DUE assignment is a specific case of the MNL-SUE assignment, the relative gap for DUE is a specific version of the relative gap for MNL-SUE. The switch between both relative gaps is the parameter \( \beta \).

• For measuring the convergence of a DUE assignment, the parameter \( \beta \) needs to be set as \( 0 \). As a result, the stochastic \( \ln(t_{rd})/E(t_{rd}) \) becomes zero and the perceived travel time \( t_{rd} \) equals the minimum travel time \( t_{od} \). Hence, all stochastic elements are filtered out of the gap function and the relative gap indicates the convergence of DUE, encoded based on travelers objectively choosing a route with minimum travel time.

• Measuring the convergence of MNL-SUE works by setting the parameter \( \beta \) to 1. As a result, the stochastic \( \ln(t_{rd})/E(t_{rd}) \) becomes zero and the minimum travel time \( t_{od} \) equals the minimum travel time \( t_{rd} \). Hence, all stochastic elements are filtered out of the gap function and the relative gap indicates the convergence of MNL-SUE, encoded based on travelers objectively choosing a route with minimum travel time. Then there occurs a shift from a measurable route travel time to subjective route utility as route choice criterion.

In summary, the relative gap (RelGap-DUE) for DUE and MNL-SUE with \( \beta = 1.0 \) measures the convergence of the assignment process. The idea is now to form a convergence measure for MNL-SUE by looking at the complementary slackness condition and the KKT multipliers in the stochastic case. For that, we consider the optimization problem for MNL-SUE by fixing \( \beta = 0.01 \).

Development of a Relative Gap for Logit-based SUE (and DUE)

Optimization problem (MNL-SUE)

\[
\begin{align*}
\min & \quad \sum_{r} \left( \sum_{d} x_{rd} \ln \left( \frac{E(t_{rd})}{t_{rd}} \right) \right) \quad \text{s.t.} \quad \sum_{d} x_{rd} = d_{r} \quad \forall r, \quad \sum_{r} x_{rd} = d_{d} \quad \forall d, \quad x_{rd} \geq 0 \quad \forall r, \quad d
\end{align*}
\]

Example

We use the two-route network as the figure below as a basis for the following numerical examples. For simplicity, each route consists of one link and we assume a linear relationship between the flow and travel time. The traffic flow for the origin-destination pair is assumed to be \( d_1 = 1000 \) and the flow conservation constraint is \( d_1 + d_2 = 1000 \) holds at all times.

Behavior of RelGap-MNL for Different β

For \( \beta = 0.01 \), the graph of RelGap-MNL (Figure 4b) behaves similarly to the graph of RelGap-DUE (Figure 4a). This concurs with our suggestion to use RelGap-MNL with \( \beta = 0.01 \) for a DUE assignment. However, if we decrease \( \beta \) towards zero, the behavior of the graph of RelGap-MNL changes as the minimum point moves along the x-axis, increasing the traffic flow on route 1 towards a new optimal solution of MNL-SUE traffic flows.

In summary, the relative gap (RelGap-DUE) for DUE and MNL-SUE with \( \beta = 0.01 \) can be derived.